1. Dominant Balances in an Idealized Convection Zone

Consider an ideal atmosphere: "atmosphere" denoting here a density-stratified fluid layer in a gravitational field, and "ideal" meaning that no microscopic dissipation is operating to cause the diffusion of momentum or heat. Any given fluid element in the atmosphere is assumed to be in local thermodynamic equilibrium (LTE). Now, as a test of stability, imagine perturbing a bubble of the fluid upward slightly. Assuming it can equilibrate to the new local pressure as it moves, this bubble will experience an adiabatic change of volume during its rise, as it has no way to exchange heat with neighboring fluid. It may then find that its density has become lower than the density of its surroundings, and if this is the case, the bubble will continue to rise. The atmosphere is thus unstable to convection. This is the well-known Schwarzschild (1906) criterion for convective instability.

It will be useful to quantify this condition in order to establish a few notational conventions. To begin, one simply applies the chain rule:

$$\Delta\rho' = \Delta\rho'(p,s) \approx \left(\frac{\partial\rho}{\partial p}\right)_s \Delta p' + \left(\frac{\partial\rho}{\partial s}\right)_p \Delta s' \ . \tag{1}$$

The density change $\Delta\rho'$ of the bubble depends only on the change in pressure $\Delta p'$ while the change in entropy $\Delta s' = 0$ because the bubble moves adiabatically by supposition. (It is assumed that displacements are made quasistatically.) The density change $\Delta\rho$ of fluid in the bubble's vicinity, on the other hand, is expressed as

$$\Delta\rho \approx \left[\left(\frac{\partial\rho}{\partial p}\right)_s \frac{dp}{dz} + \left(\frac{\partial\rho}{\partial s}\right)_p \frac{ds}{dz}\right] \Delta z \tag{2}$$

for the vertical displacement Δz. The difference between these two density changes is

$$\Delta\rho' - \Delta\rho \approx \left(\frac{\partial\rho}{\partial p}\right)_s \left[\Delta p' - \frac{dp}{dz}\Delta z\right] - \left(\frac{\partial\rho}{\partial s}\right)_p \frac{ds}{dz}\Delta z \ . \tag{3}$$

The bubble's pressure is presumed always to match the local pressure, so the quantity in square brackets is precisely zero. If the remaining term on the right hand side is negative overall, the bubble has become less dense than its surroundings, and this is the condition for instability:

$$-\left(\frac{\partial\rho}{\partial s}\right)_p \frac{ds}{dz}\Delta z = -\frac{T}{c_p}\left(\frac{\partial\rho}{\partial T}\right)_p \frac{ds}{dz}\Delta z = \frac{\rho\delta_p}{c_p}\cdot\frac{ds}{dz}\Delta z < 0 \tag{4}$$

where T is temperature, c_p is the specific heat at constant pressure, and $\delta_p \equiv -(\partial \ln\rho/\partial \ln T)_p$ is the dimensionless thermal expansion coefficient. For a "normal" fluid, one which expands upon heating, the criterion reduces simply to $ds/dz < 0$ (where z increases upward in this notation). The instability to convection is therefore controlled by the vertical entropy gradient. The crucial approximations leading to this criterion are (*i*) entropy perturbations cannot equilibrate to their surroundings because they relax extremely slowly via microscopic diffusion processes, whereas (*ii*) pressure perturbations equilibrate instantaneously because they are communicated rapidly via sound waves.

Mixing length theory postulates a *near*-ideal atmosphere that is *almost* in static equilibrium, but not quite; *i.e.*, MLT considers the magnitudes of perturbations to an equilibrium state that is marginally (un)stable to convection. From the Schwarzschild criterion, it is clear that this reference equilibrium state must be isentropic: $ds/dz = 0$. Note that the term "isentropic" has commonly been referred to as "adiabatic", but the former is more accurate: "adiabatic" more appropriately describes presupposition (*i*) above, from which the isentropic condition follows as a consequence. The equilibrium state must in addition be hydrostatic, due to force balance. Together, the hydrostatic and isentropic conditions suffice to specify completely the thermodynamic profile of the atmosphere to lowest order. The next step is to construct the first-order MLT correction to this zero-order profile. This is done by replacing the laws of fluid mechanics with simple, algebraic relations between fluid quantities; the inverse of the "mixing length" is what takes the place of partial derivatives in the very few terms that are retained.

2. Relative Scalings of the Dynamic Variables

Having established the background configuration for the model, the next question is how to treat the perturbations. This is where mixing length theory gives guidance. The question that MLT addresses is essentially one of nonlinear physics: how "superadiabatic" does the temperature gradient need to be in order to *maintain* convection *self-consistently*, *i.e.*, how is the temperature gradient modified, once convection has established itself? To answer this, MLT exploits the paramount fact of stellar structure, namely that the heat flux generated by nuclear reactions in the core must be borne away. MLT further assumes that this effectively constant flux is carried completely by weakly superadiabatic convection, justifying the assumption *a posteriori*.

To quantify the convective heat flux, MLT begins where the Schwarzschild criterion left off with the rising bubble picture, imagining that a buoyant bubble must eventually lose its integrity, mix with its environs, and deposit its excess of entropy. This process is assumed to occur over a mixing length ℓ, so the amount of heat given off by the bubble is (per gram)

$$T\Delta s' - T\Delta s = -T\ell\frac{ds}{dz} = c_p\ell\left[\left(\frac{dT}{dz}\right)_{\rm ad} - \frac{dT}{dz}\right] \equiv c_p\ell\left(\Delta\nabla T\right) \ . \tag{5}$$

Usually one sets $\ell = aH$ where H is the pressure scale height and a is a constant of order unity. To convert the above to a flux, one simply multiplies its rough average value by the product of the local density ρ and the local average bubble speed V, obtaining

$$F_{\rm conv} = \tfrac{1}{2}\rho V c_p\ell\left(\Delta\nabla T\right) \ , \tag{6}$$

where $\ell(\Delta\nabla T)/2$ represents the mean perturbed temperature of all bubbles. Next, one estimates the typical speed of a bubble at a given depth by equating its kinetic energy to the work done on it (in an average sense) by the buoyancy force as it rises or falls through the representative distance $\ell/2$. Up to factors of order unity, one finds (*cf.* Schwarzschild

1958, Kim et al. 1996):

$$\frac{\rho V^2}{2} = -\frac{g}{2}\left[\left(\frac{d\rho}{dz}\right)_{\rm ad} - \frac{d\rho}{dz}\right]\frac{\ell^2}{4} = \frac{\rho}{T}\delta_p\left[\left(\frac{dT}{dz}\right)_{\rm ad} - \frac{dT}{dz}\right]\frac{g\ell^2}{8} = \frac{\delta_p \rho g \ell^2}{8}\frac{\Delta\nabla T}{T} \ . \tag{7}$$

Notice that for the purpose of closing the set of equations, a Boussinesq-like equality (which is exact in this case—see Clayton 1968) has been invoked: density variations are linked directly to temperature variations against a hydrostatic, pressure-equilibrated backdrop.

Equations (6) and (7) can be solved for the perturbed quantities $\Delta\nabla T$ and V, and this establishes the approximate superadiabaticity of the convection zone at any given depth. But it will be more helpful in the present context to display the well-known MLT results in a dimensionless form. The relevant dimensionless quantities are the superadiabaticity parameter $\Delta\nabla$ given by

$$\Delta\nabla = -\frac{H}{c_p}\frac{ds}{dz} = \frac{H\Delta\nabla T}{T} = -\left(\frac{d\ln T}{d\ln p}\right)_{\rm ad} + \frac{d\ln T}{d\ln p} \ , \tag{8}$$

and the Mach number $M \equiv V/c_s$. There is an important and interesting relationship between the speed of sound c_s and the pressure scale height H. First, note that by applying thermodynamic identities one can write

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s = -\left(\frac{\partial s}{\partial \rho}\right)_p\left(\frac{\partial p}{\partial s}\right)_\rho = -\frac{c_p}{c_v}\left(\frac{\partial T}{\partial \rho}\right)_p\left(\frac{\partial p}{\partial T}\right)_\rho = \gamma\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{\gamma}{\alpha_T}\left(\frac{p}{\rho}\right) \ . \tag{9}$$

If the ionization state of the gas is allowed to vary, then the ratio of specific heats γ can depend on ρ and T—as can the dimensionless isothermal compressibility $\alpha_T \equiv (\partial \ln \rho/\partial \ln p)_T$ (Lantz 1992). For an ideal gas, $\gamma = 5/3$ while $\alpha_T = \delta_p = 1$. From eq. (9) and the hydrostatic approximation the scale height H can be written:

$$H \equiv -\left(\frac{d\ln p}{dz}\right)^{-1} \approx \frac{p}{\rho g} = \frac{\alpha_T}{\gamma g}{c_s}^2 \ . \tag{10}$$

Solving for $\Delta\nabla$ and M in equations (6) and (7) and using these relationships, one finds

$$M = U_1(\Delta\nabla)^{1/2} = U_2\frac{(F_{\rm conv}/\rho)^{1/3}}{c_s} \tag{11}$$

where

$$U_1 \equiv \left[\frac{a^2\delta_p\alpha_T}{4\gamma}\right]^{1/2} \sim O(1), \qquad U_2 \equiv \left[\frac{a\delta_p r_*}{2c_p}\right]^{1/3} = \left[\frac{a\alpha_T\,(\gamma-1)}{2\delta_p\gamma}\right]^{1/3} \sim O(1)\ . \tag{12}$$

The latter assumes a gas law of the form $p = r_*\rho T$, where r_* is the ideal gas constant normalized by the average mass per particle. The general relation $c_p - c_v = r_*\delta_p^2/\alpha_T$ has been used in obtaining the second expression for U_2. Note that U_2 can also be written as $(\frac{1}{2}a\nabla_{\rm ad})^{1/3}$ where the dimensionless adiabatic temperature gradient $\nabla_{\rm ad} \equiv (\partial \ln T/\partial \ln p)_s$. Eliminating c_s from expression (11) for $M = V/c_s$, one sees that $F_{\rm conv} \sim \rho V^3$, which clearly puts $F_{\rm conv}$ on the same scale as the kinetic energy flux. All these expressions leave open the possibility that the ionization state of the working fluid (and thus the various thermodynamic coefficients like r_*) may be varying as functions of depth (Lantz 1992).

The above result holds only if such departures from the equilibrium state are small. Since $\Delta\nabla$ sets the scale for all perturbations, its magnitude governs the basic validity of MLT. Near the Sun's photosphere, all the necessary quantities can all be inferred from observation. There, $\Delta\nabla$ turns out to be only on the order of 10^{-1}. But fortunately, because of its inverse dependence on density and temperature, $\Delta\nabla$ shrinks rapidly as one goes deeper into the convecting regions of the Sun: below the level of the solar granulation, $\Delta\nabla$ should be about 10^{-2}; around the base of the supergranules, 10^{-5}; and at the bottom of the convection zone, 10^{-7}. These estimates are computed based on a standard solar model (Ulrich 1970, Bahcall and Ulrich 1988), which itself makes use of MLT; the standard solar model is in this way self-consistent. Evidently, the solar plasma becomes more and more nearly isentropic the deeper one descends into the convecting region.

To recapitulate, the need to determine the structure of stellar convection zones has led to the development of an intuitive description of convective heat transport called "mixing length theory." The velocities and perturbations estimated by MLT were shown to be sufficiently small that a stratification which is neutrally stable to convection should be a good zero-order approximation to the thermodynamics. Such a stratification was found to be hydrostatic and isentropic. Heat transport is postulated to be carried out solely by the rising and falling of buoyant and antibuoyant bubbles that move adiabatically over vertical distances on the order of a "mixing length", comparable to the local pressure scale height. Thermodynamic fluctuations arising from convection are on the same order as the (slight) departure of any quantity's vertical gradient from its stable gradient, times the mixing length. The ratio $\Delta\nabla$ of fluctuations to the background level scales with the square of the Mach number. Even though MLT cannot offer a detailed portrait of the internal structure of a convection zone, it does give a sketch of the correct order of magnitude of its properties. MLT is also consistent with computer simulations of convection, lending credence to its predictions and estimates.

3. References

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