Exercises

10.6 **Fluctuation-dissipation: Ising.**[1] (Condensed matter) ③

This exercise again needs a simulation of the Ising model; you can use one we provide in the computer exercises portion of the text web site [129].

Let us consider the Ising Hamiltonian in a time-dependent external field $H(t)$,

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j - H(t) \sum_i S_i, \qquad (1)$$

and look at the fluctuations and response of the time-dependent magnetization $M(t) = \sum_i S_i(t)$. The Ising model simulation should output both the time-dependent magnetization per spin $m(t) = (1/N) \sum_i S_i$ and the time–time correlation function of the magnetization per spin,

$$c(t) = \Big\langle (m(0) - \langle m \rangle_{\mathrm{eq}})(m(t) - \langle m \rangle_{\mathrm{eq}}) \Big\rangle_{\mathrm{ev}}. \qquad (2)$$

We will be working above T_c, so $\langle m \rangle_{\mathrm{eq}} = 0$.[2]

The time–time correlation function will start non-zero, and should die to zero over time. Suppose we start with a non-zero small external field, and turn it off at $t = 0$, so $H(t) = H_0 \Theta(-t)$.[3] The magnetization $m(t)$ will be non-zero at $t = 0$, but will decrease to zero over time. By the Onsager regression hypothesis, $m(t)$ and $c(t)$ should decay with the same law.

Run the Ising model, changing the size to 200×200. Equilibrate at $T = 3$ and $H = 0$, then do a good measurement of the time–time auto-correlation function and store the resulting graph. (Rescale it to focus on the short times before it equilibrates.) Now equilibrate at $T = 3$, $H = 0.05$, set $H = 0$, and run for a short time, measuring $m(t)$.

(a) *Does the shape and the time scale of the magnetization decay look the same as that of the auto-correlation function? Measure $c(0)$ and $m(0)$ and deduce the system-scale $C(0)$ and $M(0)$.*

Response functions and the fluctuation-dissipation theorem. The response function $\chi(t)$ gives the change in magnetization due to an infinitesimal impulse in the external field H. By superposition, we can use $\chi(t)$ to generate the linear response to any external perturbation. If we impose a small time-dependent external field $H(t)$, the average magnetization is

$$M(t) - \langle M \rangle_{\mathrm{eq}} = \int_{-\infty}^{t} \mathrm{d}t' \; \chi(t - t') H(t'), \qquad (10.93)$$

where $\langle M \rangle_{\mathrm{eq}}$ is the equilibrium magnetization without the extra field $H(t)$ (zero for us, above T_c).

(b) *Using eqn 10.93, write $M(t)$ for the step down $H(t) = H_0 \Theta(-t)$, in terms of $\chi(t)$.*

The fluctuation-dissipation theorem states

$$\chi(t) = -\beta \, \mathrm{d}C(t)/\mathrm{d}t, \qquad (10.94)$$

where $C(t) = \langle (M(0) - \langle M \rangle_{\mathrm{eq}})(M(t) - \langle M \rangle_{\mathrm{eq}}) \rangle_{\mathrm{ev}}$.

(c) *Use eqn 10.94 and your answer to part (b) to predict the relationship between the demagnetization $M(t)$ and the correlation $C(t)$ you measured in part (a). How does your analytical ratio compare with the $t = 0$ ratio you noted down in part (a)?*

[1] From *Statistical Mechanics: Entropy, Order Parameters, and Complexity* by James P. Sethna, copyright Oxford University Press, 2007, page 237. A pdf of the text is available at pages.physics.cornell.edu/sethna/StatMech/ (select the picture of the text). Hyperlinks from this exercise into the text will work if the latter PDF is downloaded into the same directory/folder as this PDF.

[2] Note that the formulæ in the text are in terms of the total magnetization $M = Nm$ and its correlation function $C = N^2 c$.

[3] Here Θ is the Heaviside function: $\Theta(t) = 0$ for $t < 0$, and $\Theta(t) = 1$ for $t > 0$.