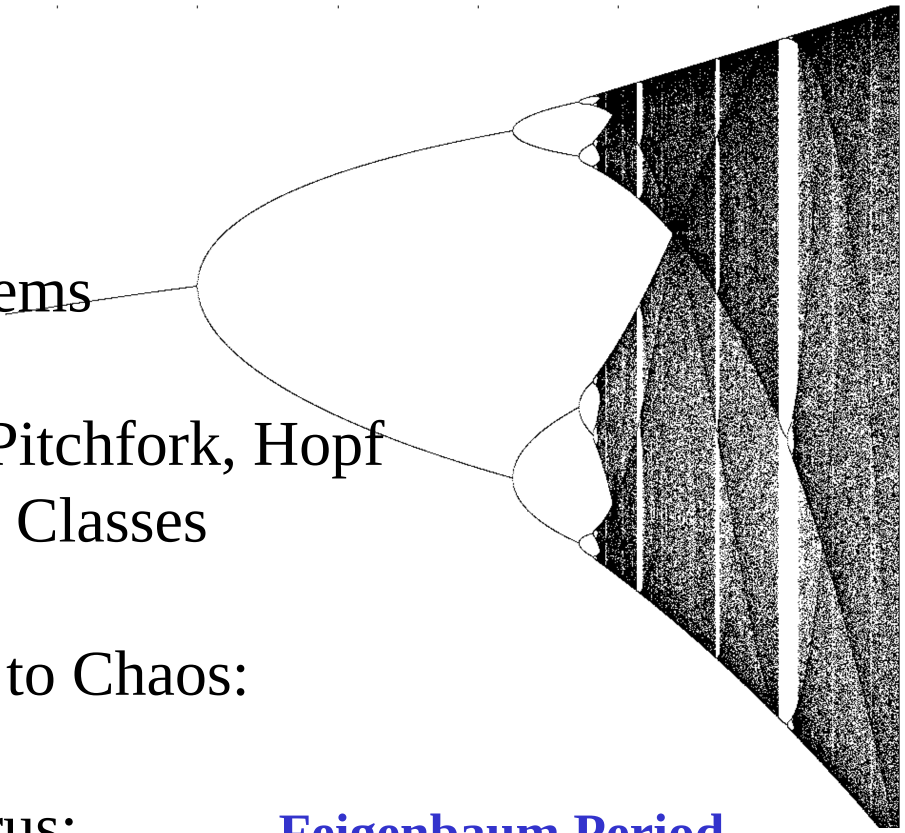
Dynamical Systems and Chaos

Coarse-Graining in Time



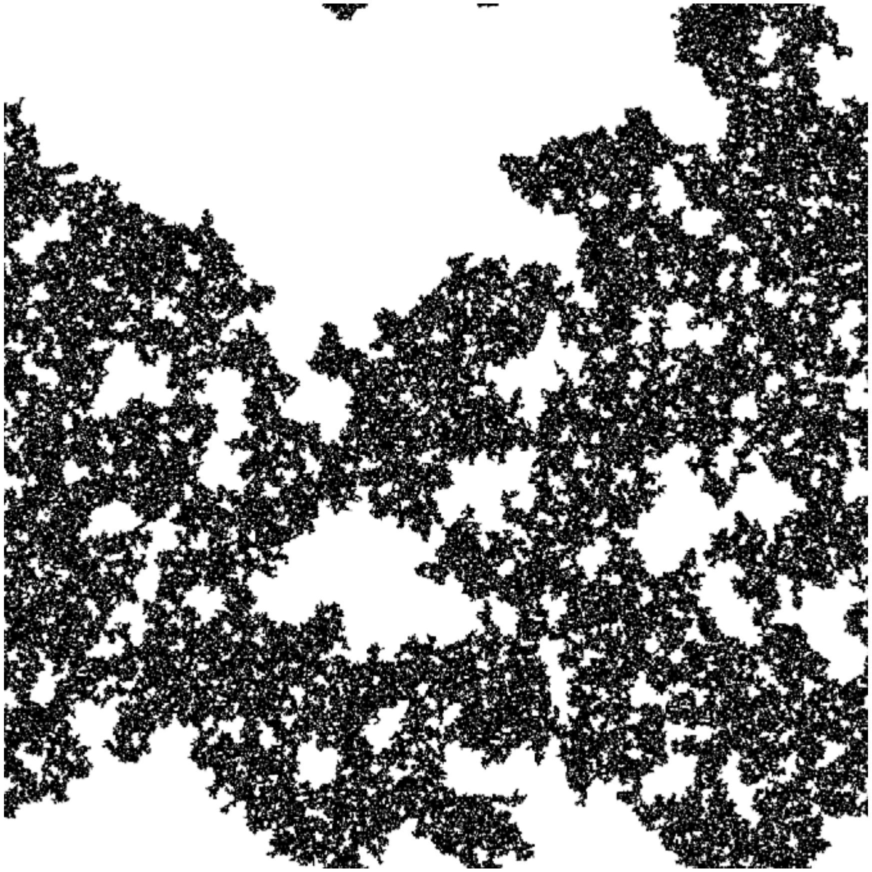
Low Dimensional Dynamical Systems

- Bifurcation Theory
    - Saddle-Node, Intermittency, Pitchfork, Hopf
    - Normal Forms = Universality Classes
- Feigenbaum Period Doubling
- Transition from Quasiperiodicity to Chaos: Circle Maps
- Breakdown of the Last KAM Torus: Synchrotrons and the Solar System

Feigenbaum Period Doubling
Attractor vs. λ
Onset of Chaos = Fractal

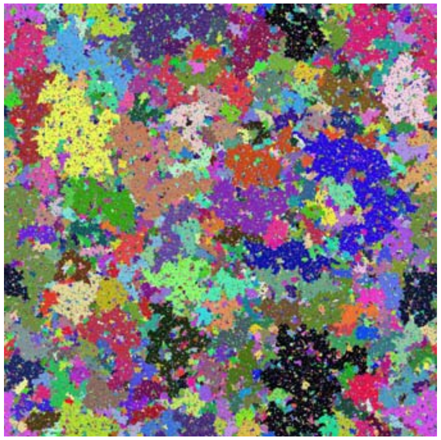
Percolation

Structure on All Scales

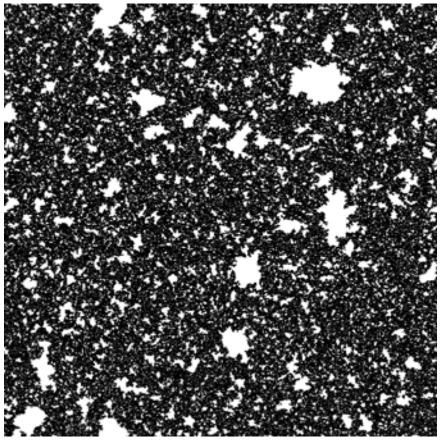


Largest Connected Cluster
$P=P_c$

- Connectivity Transition
- Punch Holes at Random, Probability 1-P
  P_c =1/2 Falls Apart
  (2D, Square Lattice, Bond)
- Static (Quenched) Disorder



P=0.49



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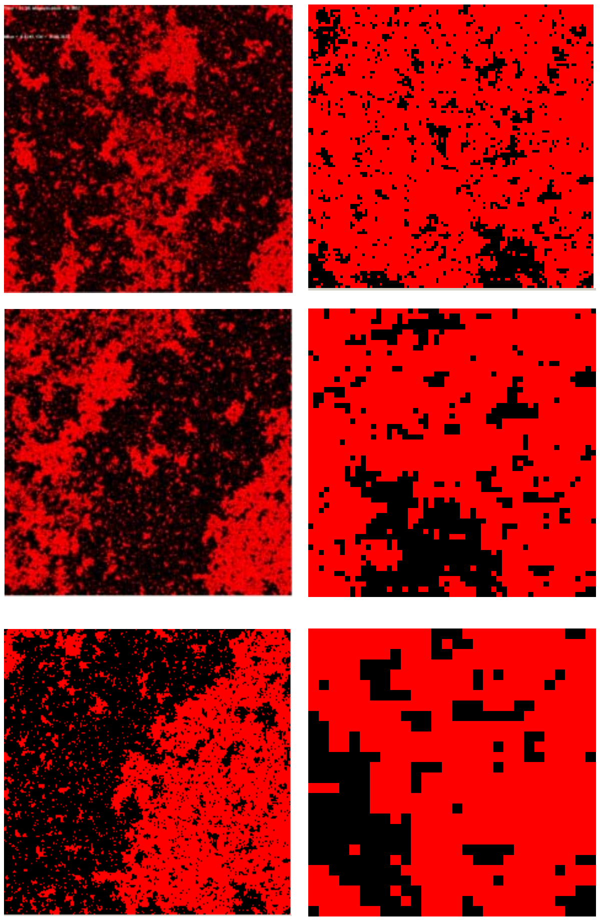
Self-Similarity

Self-Universality on Different Scales

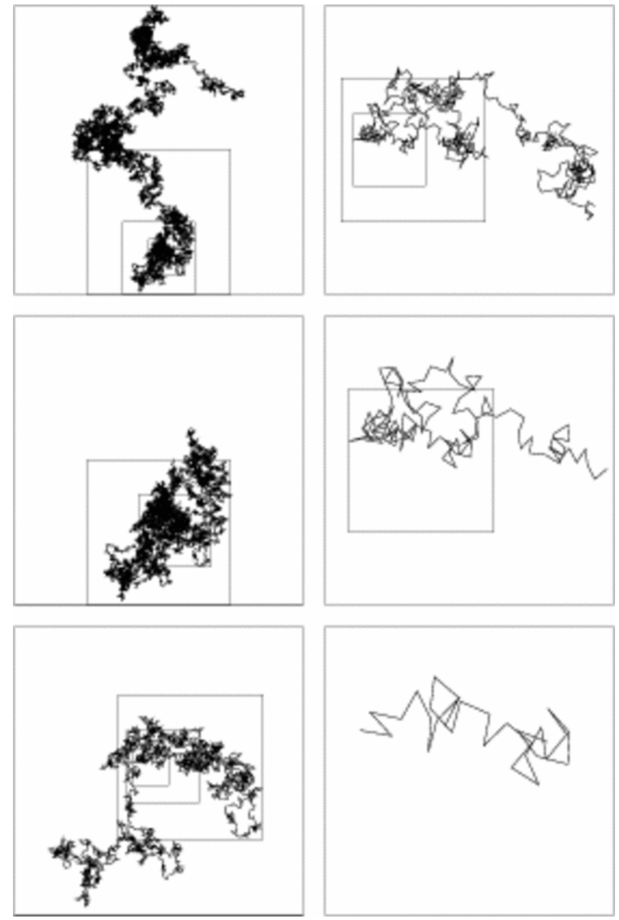


Ising Model at T_c

Self-similarity → Power Laws

Expand rulers by
$B=(1+\varepsilon)$;
Up-spin cluster size S, probability distribution D(S)

$D[S] = A\, D[S'/C]$
$=(1+a\varepsilon)\, D[(1+c\varepsilon)\, S')]$
$a\, D = -cS'\, dD/dS$
$D[S] = D_0\, S^{-a/c}$



Random Walks

Universal critical exponents $c=d_f=1/\sigma\nu$, $a/c=\tau$: D_0 system dependent

Ising Correlation $C(x) \sim x^{-(d-2+\eta)}$ at T_c, random walk $x \sim t^{1/2}$

Fractal Dimension D_f

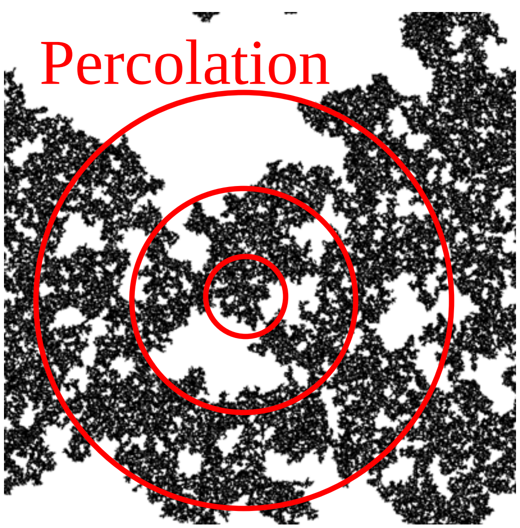
Mass ~ Size^{D_f}

Cantor Set

Middle third

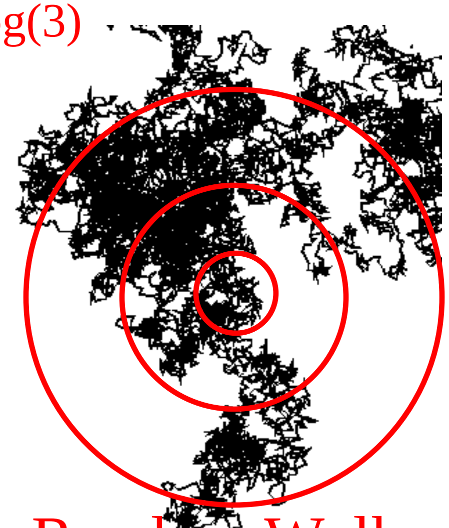
Base 3 without 1's

$D_f = \log(2)/\log(3)$

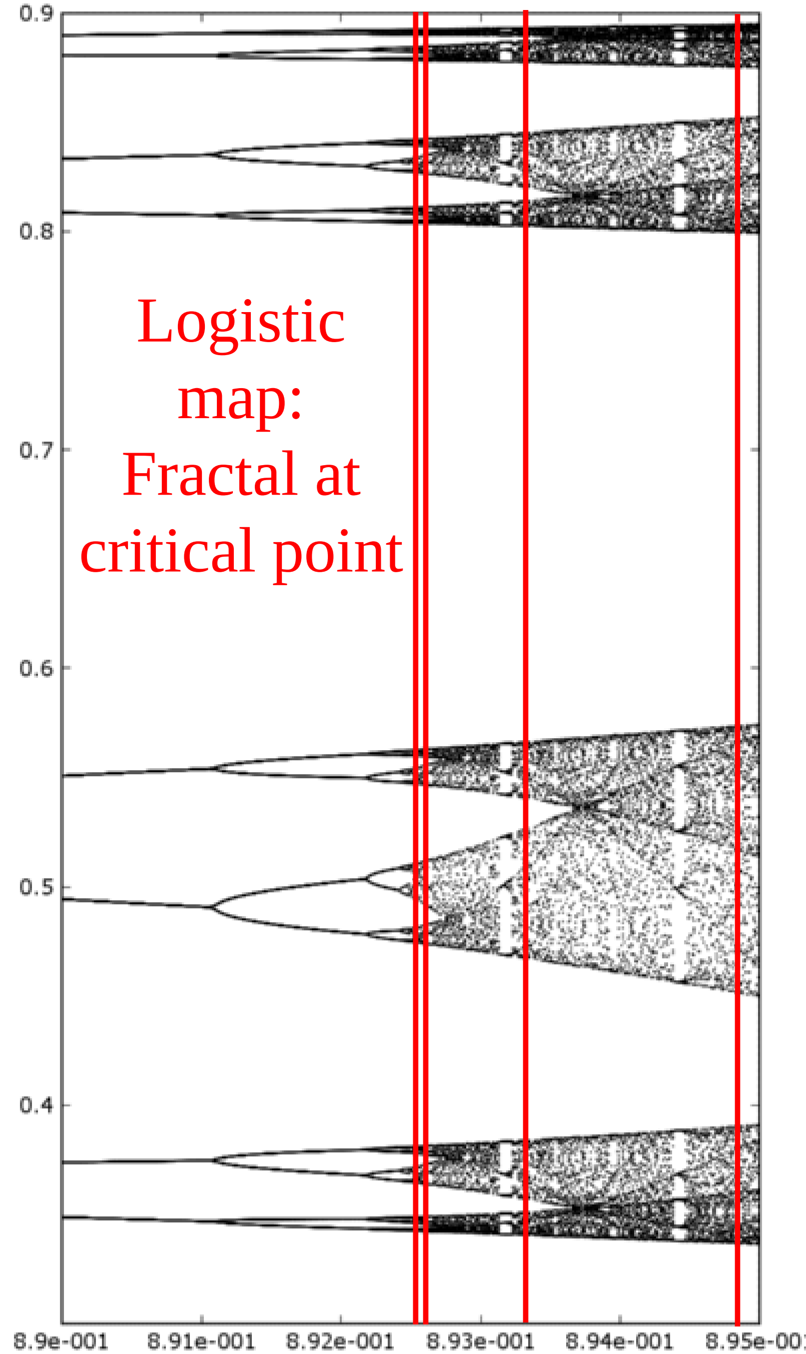


Percolation

critical point

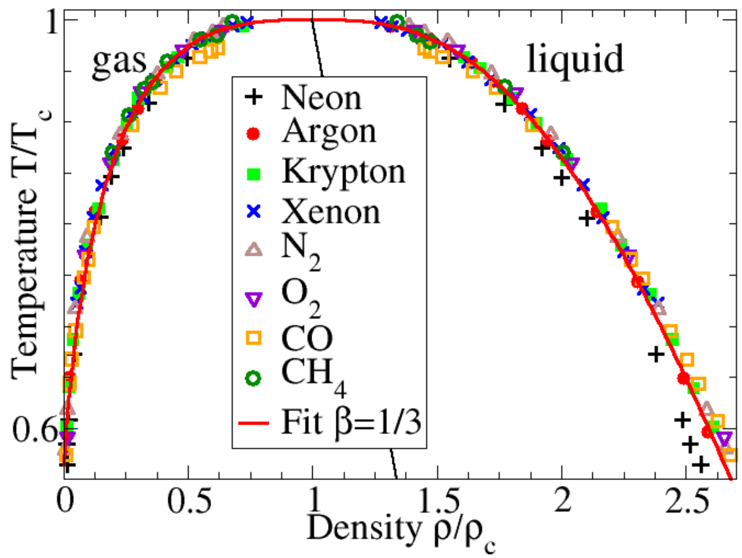


Random Walk: generic scale invariance

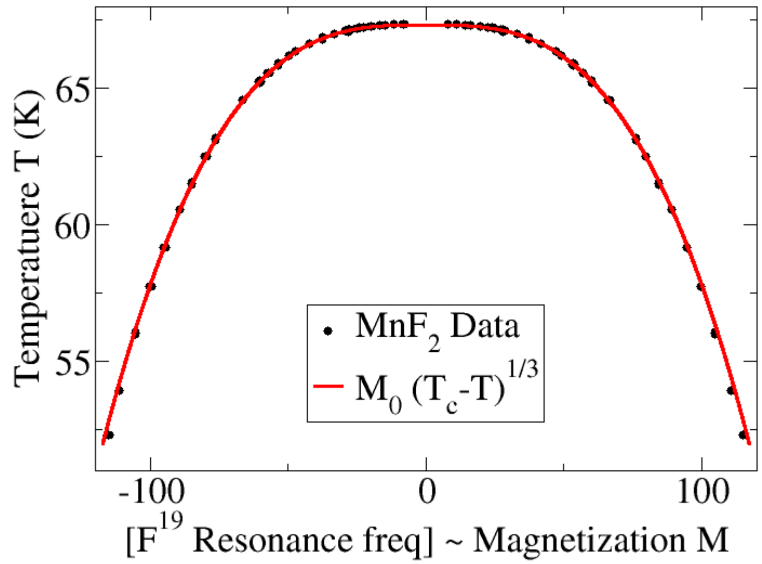


Universality: Shared Critical Behavior

Ising Model and Liquid-Gas Critical Point



Same critical exponent β=0.332!



Liquid-Gas Critical Point

$\rho - \rho_c \sim (T_c - T)^\beta$

$\rho^{Ar}(T) = A\, \rho^{CO}(BT)$

Ising Critical Point

$M(T) \sim (T_c - T)^\beta$

$\rho^{Ar}(T) = A(M(BT), T)$

Universality: Same Behavior up to Change in Coordinates

$A(M,T) = a_1 M + a_2 + a_3 T$

Nonanalytic behavior at critical point (not parabolic top)

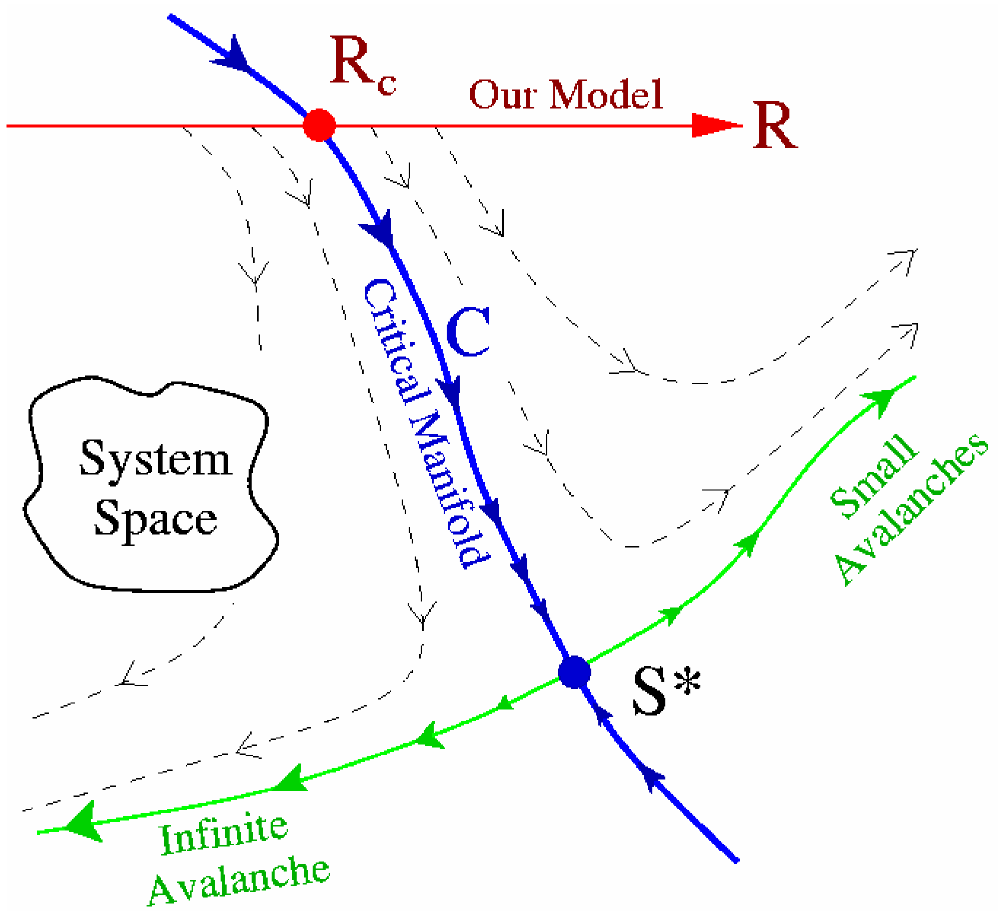
All power-law singularities (χ, c_v, ξ) are shared by magnets, liquid/gas

The Renormalization Group

Why Universal? Fixed Point under Coarse Graining

Renormalization Group

- Not a group
- *Renormalized* parameters
  (electron charge from QED)
- Effect of coarse-graining
  (shrink system, remove
  short length DOF)
- Fixed point **S*** ***self-similar***
  (coarse-grains to self)
- Critical points flow to **S***
  Universality
- Many methods (technical)
  real-space, ε-expansion, Monte Carlo, …
- Critical exponents from
  linearization near fixed point



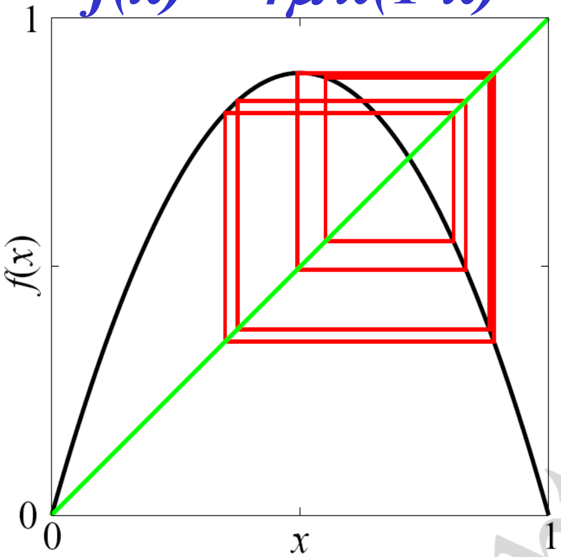
System Space Flows
Under Coarse-Graining

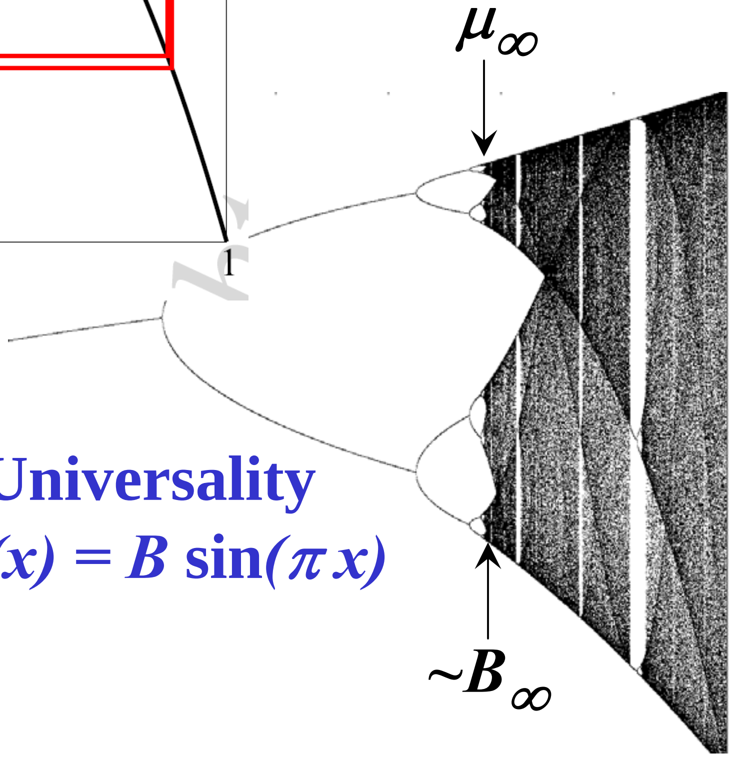
Renormalization Group

Coarse-Graining in Time

Dynamics = Map

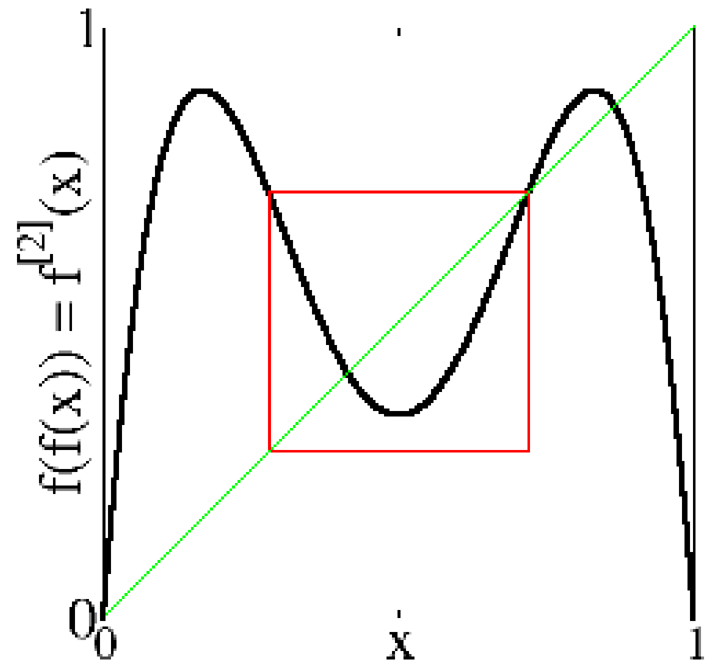
$f(x) = 4\mu\, x(1-x)$





Universality

$f_{sin}(x) = B \sin(\pi x)$



Renormalization Group

$x_n = f(x_{n-1})$

$x_0, x_1, x_2, x_3, x_4, x_5, \ldots$

New map: $x_n = f(f(x_{n-2}))$

$x_0, x_2, x_4, x_6, x_8, x_{10}, \ldots$

Decimation by two!