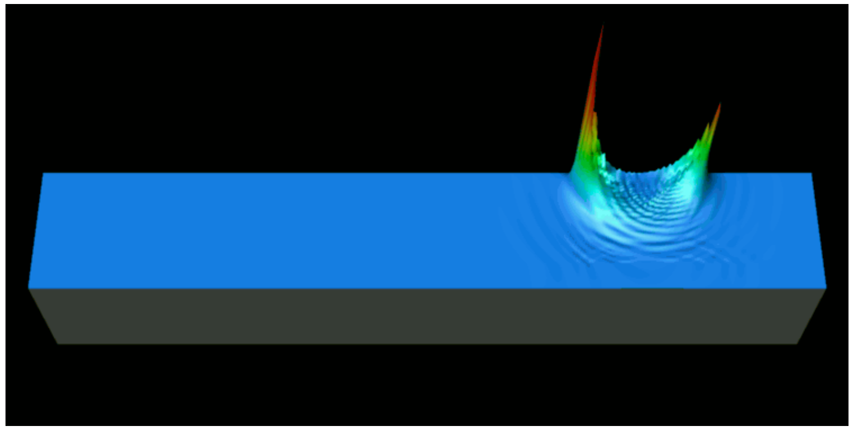
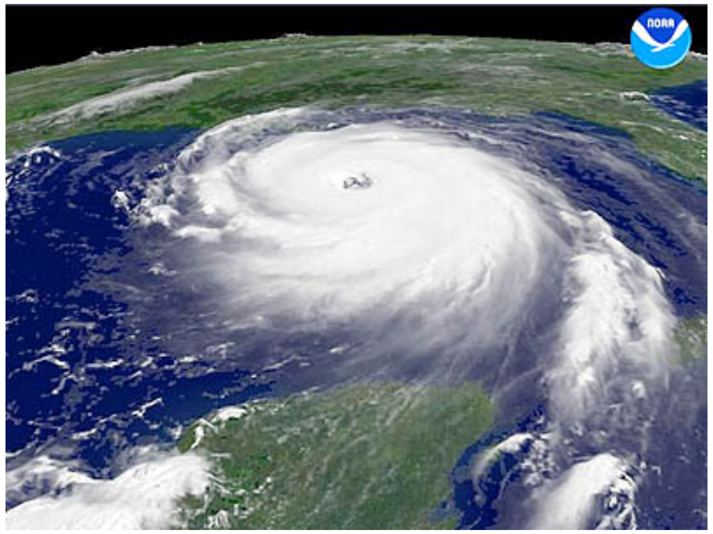
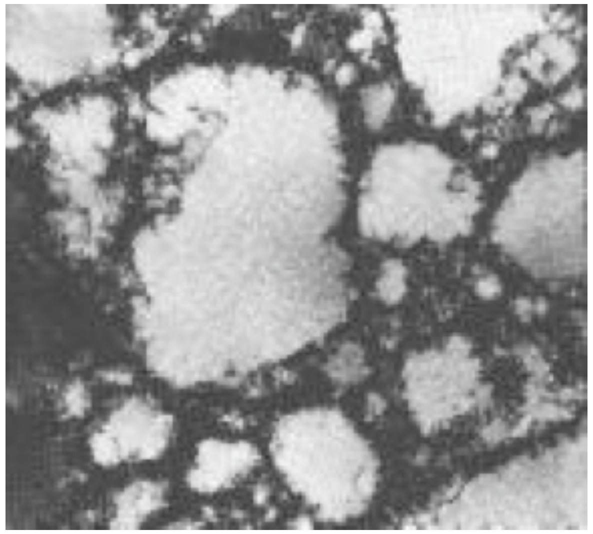
Pattern formation, cardiac dynamics, and spiral waves

Phys 7682 / CIS 6229: Computational Methods for Nonlinear Systems

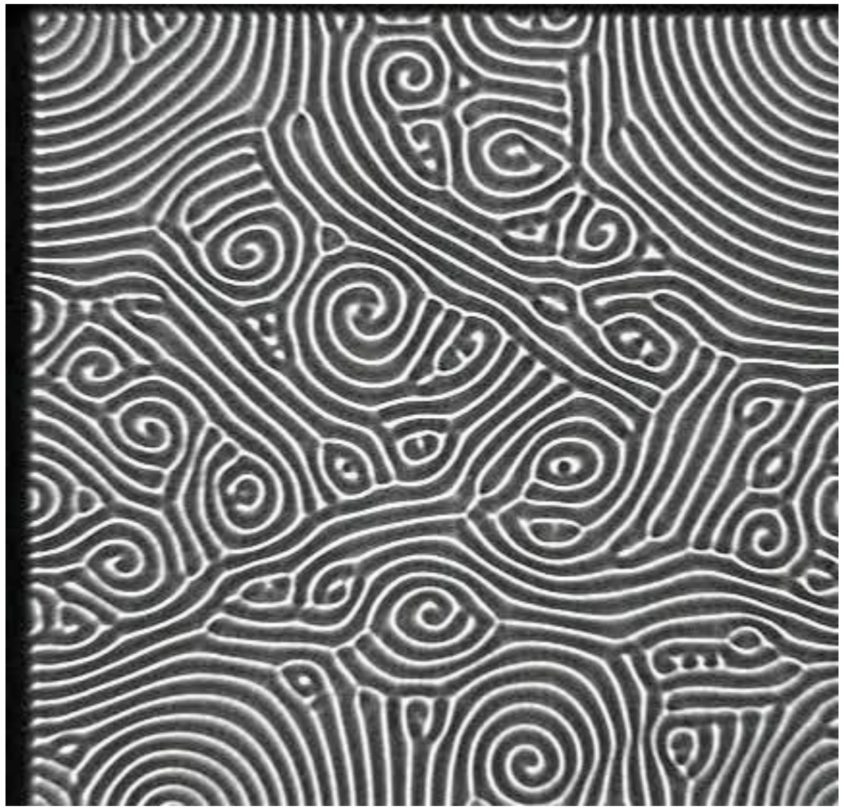
- patterns are ubiquitous in spatiotemporal systems driven out of equilibrium
  - regular, periodic patterns
  - localized, coherent structures ("defects")



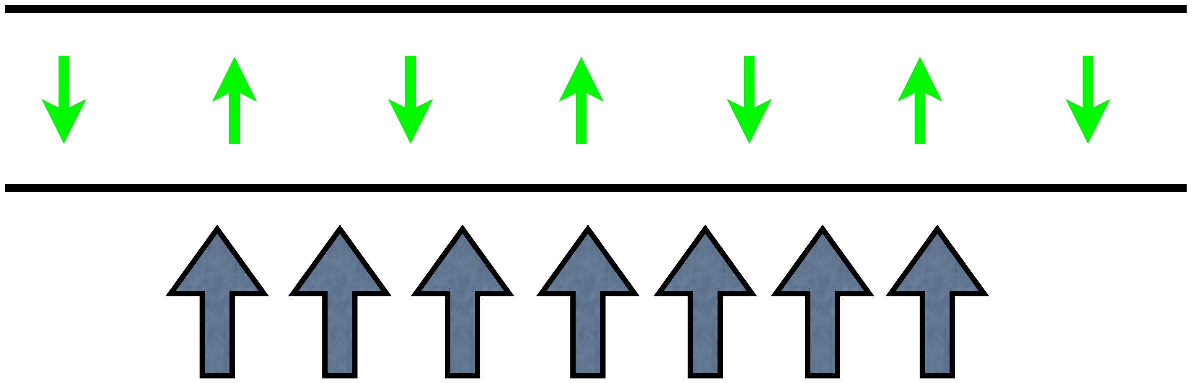
The universality of patterns*

FitzHugh-Nagumo model



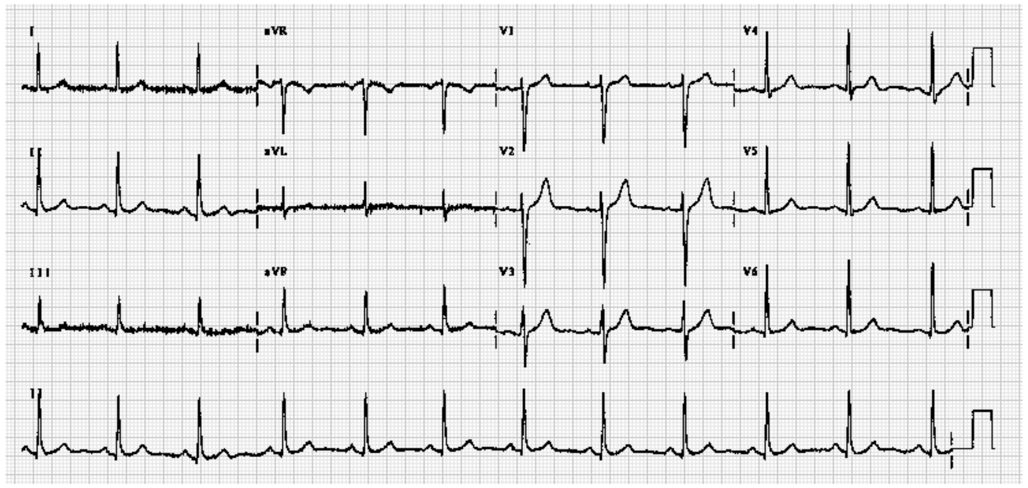
Rayleigh-Benard convection
(Bodenschatz)

cAMP spiral waves in
Dictyostelium chemotaxis
from hopf.brandeis.chem.edu

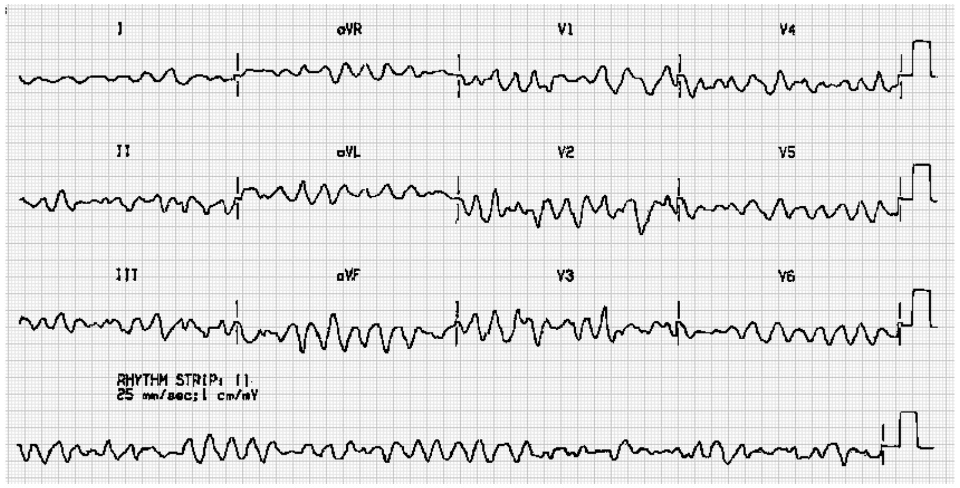


*see, e.g.,
Cross & Hohenberg,
Rev. Mod. Phys. 65(3), 1993

Cardiac dynamics



normal electrocardiogram (ECG)



ecglibrary.com

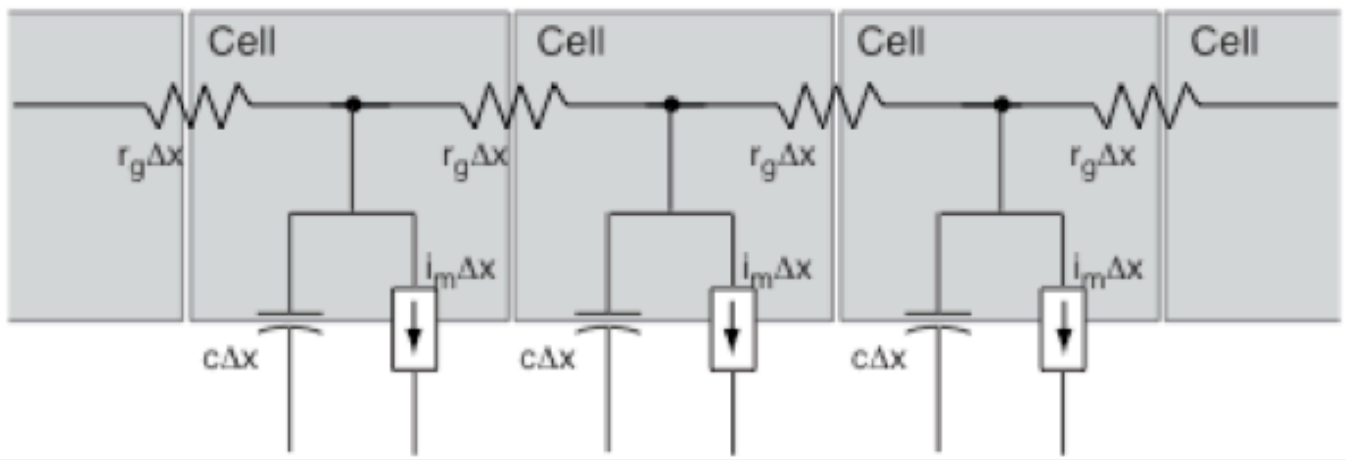
ECG during ventricular fibrillation

spiral waves implicated in arrhythmia:
incoherent pumping due to many different local pulse sources

FitzHugh-Nagumo model

- simple model of transmembrane voltages and currents in biological tissue
- reduction of Hodgkin-Huxley model of nerve conduction to two state variables

$$\frac{\partial V}{\partial t} = \nabla^2 V + \frac{1}{\epsilon}(V - V^3/3 - W)$$
$$\frac{\partial W}{\partial t} = \epsilon(V - \gamma W + \beta),$$



Otani

↑ nonlinear response within cells

⟵ diffusive coupling across cells ⟶

V = transmembrane voltage

W = recovery variable

Cardiac tissue is an *excitable medium*

reaction-diffusion equation

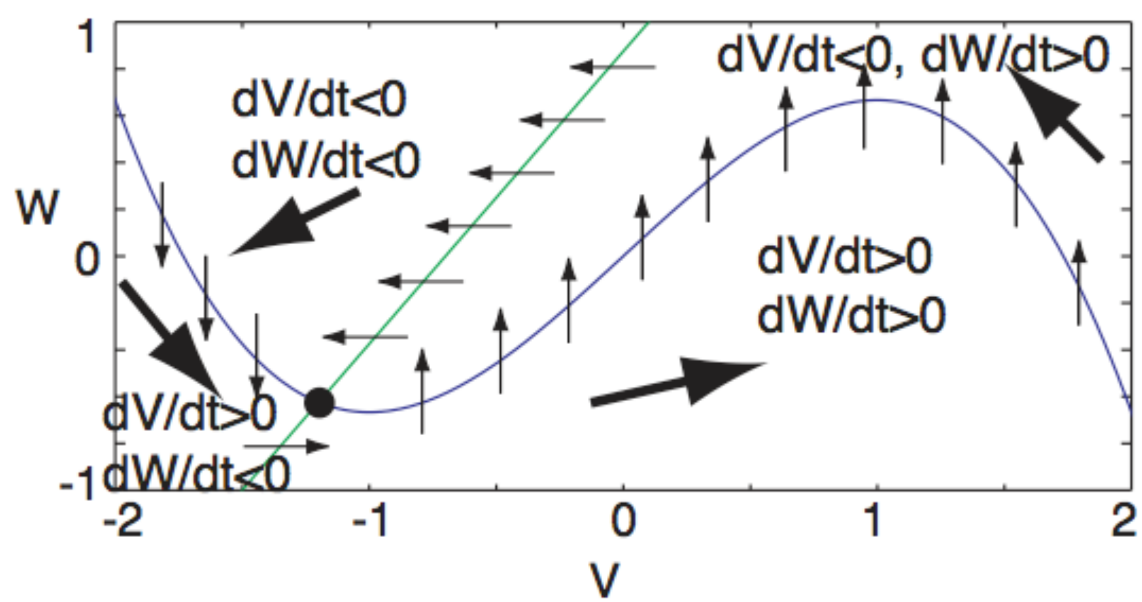
Spatially-independent FitzHugh-Nagumo model

$$\frac{\partial V}{\partial t} = \nabla^2 V + \frac{1}{\epsilon}(V - V^3/3 - W)$$
$$\frac{\partial W}{\partial t} = \epsilon(V - \gamma W + \beta),$$

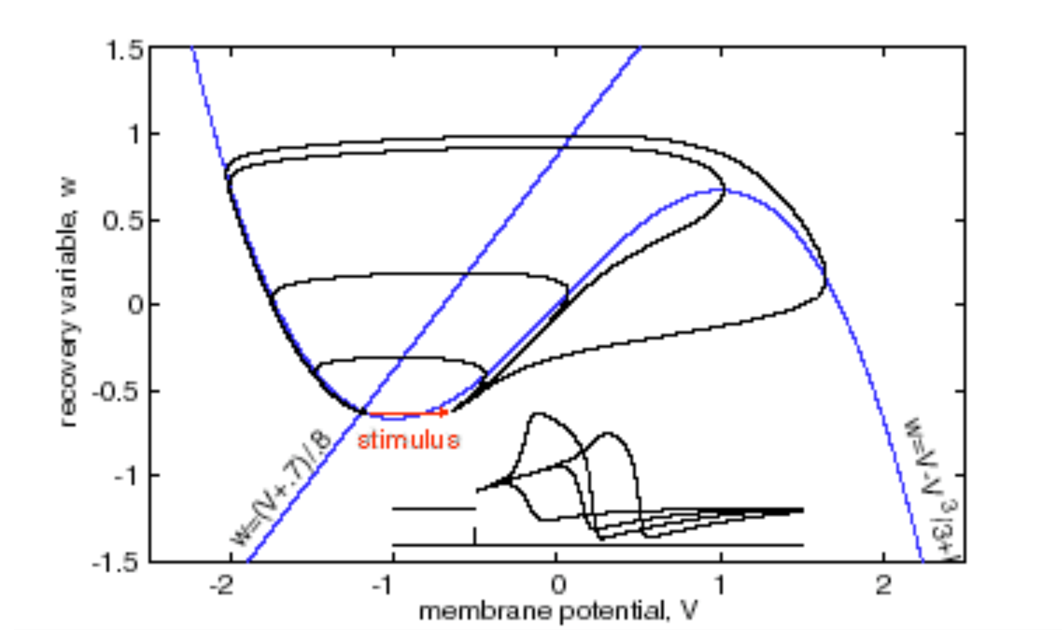
Nullcline analysis

V-nullcline: curve on which dV/dt = 0
W-nullcline: curve on which dW/dt = 0

→ use root-finder to find point (V*, W*) at which (dV/dt, dW/dt) = 0



Otani



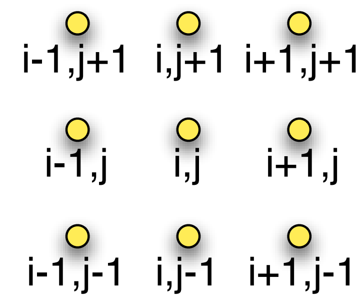
Scholarpedia

voltage pulses followed by refractory period

Two-dimensional FitzHugh-Nagumo model

$$\frac{\partial V}{\partial t} = \nabla^2 V + \frac{1}{\epsilon}(V - V^3/3 - W)$$

$$\frac{\partial W}{\partial t} = \epsilon(V - \gamma W + \beta)$$



Finite-difference approximations

$$\frac{\partial^2 V}{\partial x^2} \approx (V(x+dx,y) - 2V(x,y) + V(x-dx,y))/dx^2$$

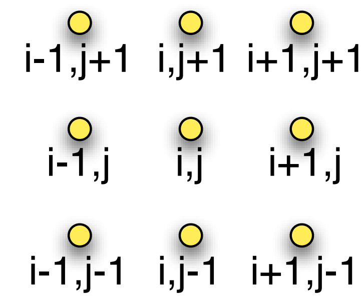
$$\frac{\partial^2 V}{\partial y^2} \approx (V(x,y+dy) - 2V(x,y) + V(x,y-dy))/dy^2$$

$$\nabla^2 V_{i,j} \approx (V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j})/dx^2$$

$$\nabla^2 V_{i,j} \approx \frac{1}{dx^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Stencils

$$\nabla^2 V_{i,j} \approx \frac{1}{dx^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

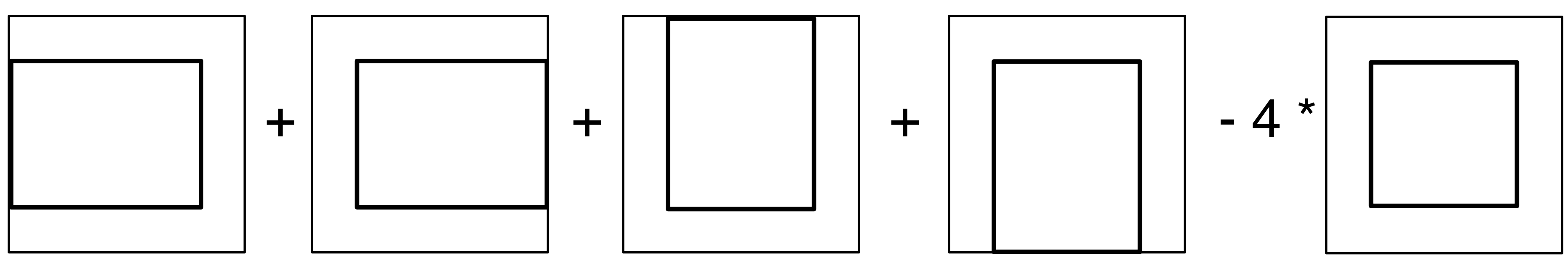


$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

instead of :

```
for i in range(1,Nx-1):
   for j in range(1,Ny-1):
      d2V[i,j] = V[i+1,j]-V[i-1,j]+V[i,j+1]+V[i,j-1]-4*V[i,j]
```

use array operations to overlay shifted copies of array



stencils not unique
(same order,
better fidelity)

$$\nabla^2 V_{i,j} \approx \frac{1}{dx^2}\begin{pmatrix} 1/6 & 2/3 & 1/6 \\ 2/3 & -10/3 & 2/3 \\ 1/6 & 2/3 & 1/6 \end{pmatrix}$$

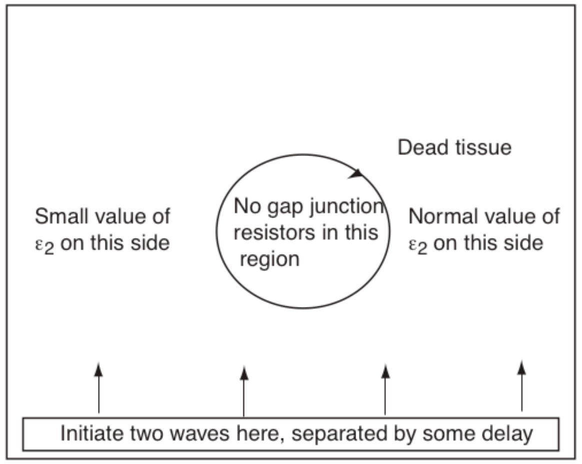
Extensions of the basic model

Niels F. Otani, Further exploration of the FitzHugh-Nagumo model (Project Topics, Section 5, p. 24)

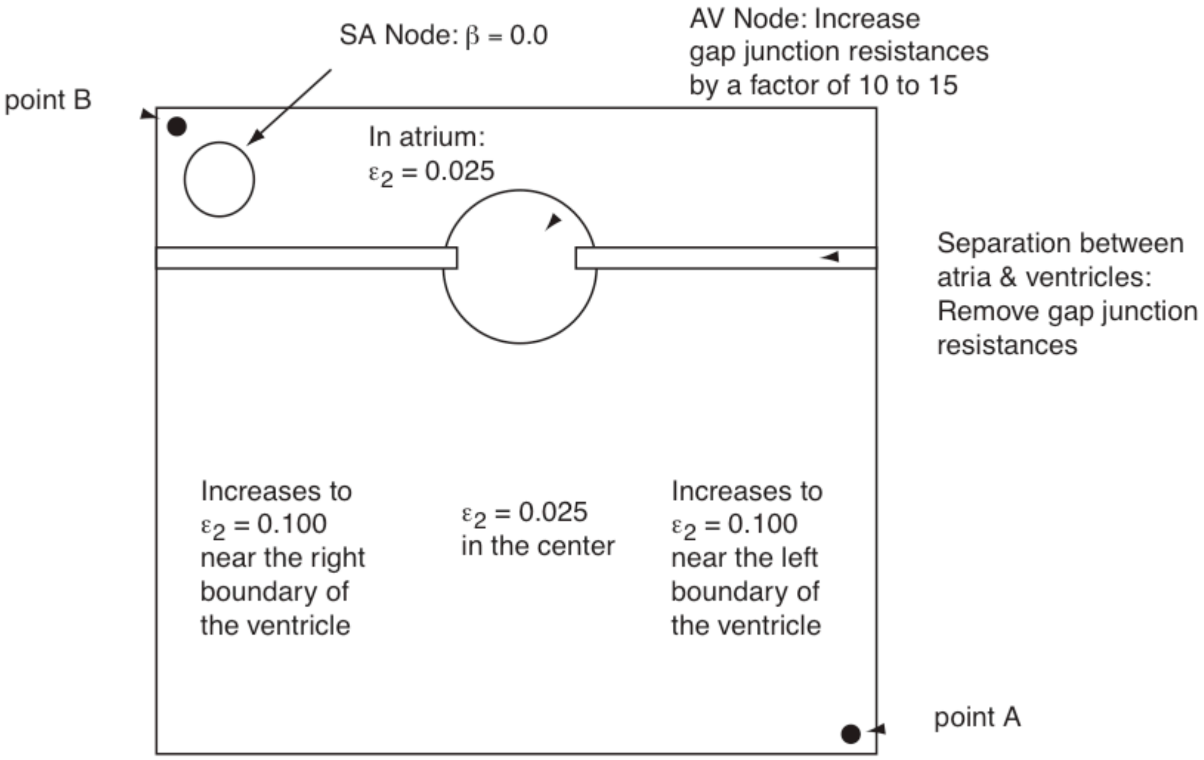
$$\frac{\partial V}{\partial t} = \nabla^2 V + \frac{1}{\epsilon}(V - V^3/3 - W)$$

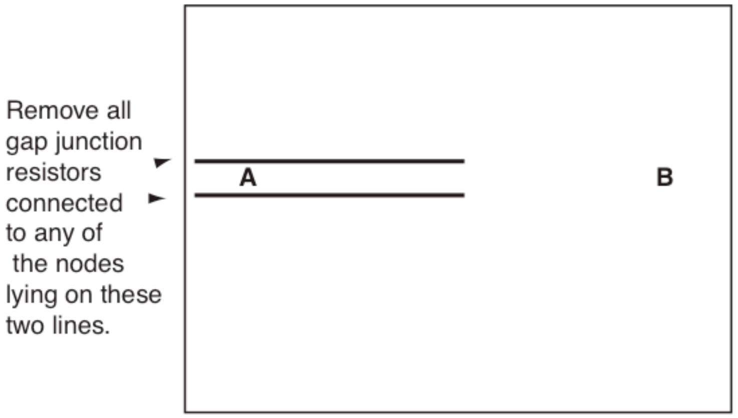
$$\frac{\partial W}{\partial t} = \epsilon(V - \gamma W + \beta)$$

spatially-varying parameters
(inc. diffusive coupling D)



Dead tissue





Two-chamber geometry with pacemaker
(sino-atrial node)

Source-sink characteristics