Exercises

2.5 **Generating random walks.**[1][2] (Computation) ③

One can efficiently generate and analyze random walks on the computer.

(a) *Write a routine to generate an N-step random walk in d dimensions, with each step uniformly distributed in the range $(-1/2,\ 1/2)$ in each dimension. (Generate the steps first as an $N \times d$ array, then do a cumulative sum.) Plot x_t versus t for a few 10 000-step random walks. Plot x versus y for a few two-dimensional random walks, with N = 10, 1000, and 100 000. (Try to keep the aspect ratio of the XY plot equal to one.) Does multiplying the number of steps by one hundred roughly increase the net distance by ten?*

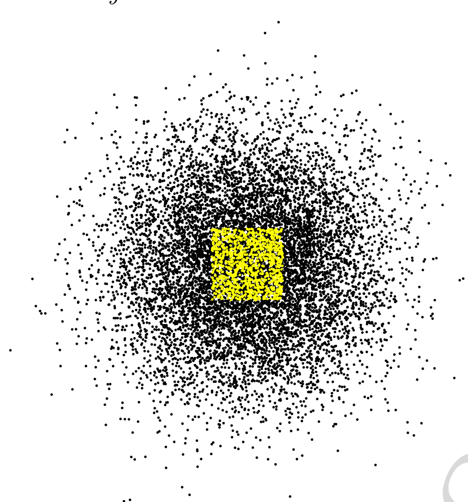

Fig. 2.10 Emergent rotational symmetry. Endpoints of many random walks, with one step (central square of bright dots) and ten steps (surrounding pattern). Even though the individual steps in a random walk break rotational symmetry (the steps are longer along the diagonals), multi-step random walks are spherically symmetric. The rotational symmetry emerges as the number of steps grows.

Each random walk is different and unpredictable, but the *ensemble* of random walks has elegant, predictable properties.

(b) *Write a routine to calculate the endpoints of W random walks with N steps each in d dimensions. Do a scatter plot of the endpoints of 10 000 random walks with N = 1 and 10, superimposed on the same plot. Notice that the longer random walks are distributed in a circularly symmetric pattern, even though the single step random walk $N = 1$ has a square probability distribution (Fig. 2.10).*

This is an *emergent symmetry*; even though the walker steps longer distances along the diagonals of a square, a random walk several steps long has nearly perfect rotational symmetry.[3]

The most useful property of random walks is the *central limit theorem*. The endpoints of an ensemble of N step one-dimensional random walks with root-mean-square (RMS) step-size a has a Gaussian or normal probability distribution as $N \to \infty$,

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-x^2/2\sigma^2), \qquad (2.35)$$

with $\sigma = \sqrt{N}a$.

(c) *Calculate the RMS step-size a for one-dimensional steps uniformly distributed in $(-1/2,\ 1/2)$. Write a routine that plots a histogram of the endpoints of W one-dimensional random walks with N steps and 50 bins, along with the prediction of eqn 2.35 for x in $(-3\sigma, 3\sigma)$. Do a histogram with W = 10 000 and N = 1, 2, 3,*

[1] From *Statistical Mechanics: Entropy, Order Parameters, and Complexity* by James P. Sethna, copyright Oxford University Press, 2007, page 28. A pdf of the text is available at pages.physics.cornell.edu/sethna/StatMech/ (select the picture of the text). Hyperlinks from this exercise into the text will work if the latter PDF is downloaded into the same directory/folder as this PDF.

[2] This exercise and the associated software were developed in collaboration with Christopher Myers.

[3] The square asymmetry is an *irrelevant perturbation* on long length and time scales (Chapter 12). Had we kept terms up to fourth order in gradients in the diffusion equation $\partial\rho/\partial t = D\nabla^2\rho + E\nabla^2\left(\nabla^2\rho\right) + F\left(\partial^4\rho/\partial x^4 + \partial^4\rho/\partial y^4\right)$, then F is square symmetric but not isotropic. It will have a typical size $\Delta t/a^4$, so is tiny on scales large compared to a.

and 5. How quickly does the Gaussian distribution become a good approximation to the random walk?